

# A modified mean curvature flow in Euclidean space and soap bubbles in symmetric spaces

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## Abstract

In this paper, we show that small spherical soap bubbles in irreducible simply connected symmetric spaces of rank greater than one are constructed from the limits of a certain kind of modified mean curvature flows starting from small spheres in the Euclidean space of dimension equal to the rank of the symmetric space, where we note that the small spherical soap bubbles are invariant under the isotropy subgroup action of the isometry group of the symmetric space. Furthermore, we investigate the shape and the mean curvature of the small spherical soap bubbles.

## 1 Introduction

Let  $f$  be an immersion of an  $n$ -dimensional compact oriented manifold  $M$  into an  $(n+1)$ -dimensional oriented Riemannian manifold  $(\widehat{M}, \widehat{g})$ . If  $f$  is of constant mean curvature, then  $f(M)$  is called a *soap bubble*. Soap bubbles in the Euclidean space have studied by many geometers. In 1989, W.T. Hsiang and W.Y. Hsiang ([HH]) studied isoperimetric soap bubbles in the product space  $H^{n_1}(c_1) \times \cdots \times H^{n_k}(c_k)$  of hyperbolic spaces or  $H^{n_1}(c_1) \times \cdots \times H^{n_k}(c_k) \times \mathbb{R}^{n_{k+1}}$ , where “isoperimetric” means that the soap bubble is a solution of the isoperimetric problem. They proved that all isoperimetric soap bubbles in the product spaces are invariant under the isotropy subgroup action of the isometry group of the product space. Also, they ([HH]) proved that isoperimetric soap bubbles with the same constant mean curvature in  $H^{n_1}(c_1) \times \mathbb{R}$  are congruent. Furthermore, they found the lower bound of the constant mean curvatures of isoperimetric soap bubbles in  $H^{n_1}(c_1) \times H^{n_2}(c_2)$  or  $H^{n_1}(c_1) \times \mathbb{R}^{n_2}$ . In 1992, W.Y. Hsiang ([Hs]) found the lower bound of the constant mean curvatures of (not necessarily isoperimetric) soap bubbles in a rank  $l(\geq 2)$  symmetric space  $G/K$  of non-compact type and, in the case where  $G/K$  is irreducible, he gave the explicit

description of the lower bound by using the root system of  $G/K$ . Furthermore, he proved that, for each real number  $b$  greater than the lower bound, there exists a  $K$ -invariant spherical soap bubble of constant mean curvature  $b$  in  $G/K$ .

In this paper, we introduce the notion of a weighted root system and define the modified mean curvature flow in a Euclidean space associated to the system. We show that the flows starting from small spheres exist in infinite time and converge to an embedded hypersurfaces in  $C^\infty$ -topology. Furthermore, in the case where the system is one associated to an irreducible simply connected symmetric space of rank greater than one, we show that spherical soap bubbles in the symmetric spaces are constructed from the limit hypersurfaces of the flows starting from small spheres and investigate the shape and the mean curvature of the spherical soap bubbles.

First we shall introduce the notion of a weighted root system. Let  $\mathcal{S} = (V, (\Delta, \{m_\alpha \mid \alpha \in \Delta\}, \varepsilon))$  be a system satisfying the following four conditions:

- (i)  $V$  is a  $l(\geq 2)$ -dimensional real vector space equipped with an inner product  $\langle \cdot, \cdot \rangle$ ,
- (ii)  $\Delta$  is a subset of the dual space  $V^*$  of  $V$  and it is a root system of rank  $l$  in the sense of [He] (i.e., it is of type  $\mathfrak{a}_l, \mathfrak{b}_l, \mathfrak{c}_l, \mathfrak{d}_l, \mathfrak{b}\mathfrak{d}_l, \mathfrak{e}_6$  (in case of  $l = 6$ ),  $\mathfrak{e}_7$  (in case of  $l = 7$ ),  $\mathfrak{e}_8$  (in case of  $l = 8$ ),  $\mathfrak{f}_4$  (in case of  $l = 4$ ),  $\mathfrak{g}_2$  (in case of  $l = 2$ )),
- (iii)  $m_\alpha$  ( $\alpha \in \Delta$ ) are positive integers and  $\varepsilon$  is equal to 1 or  $-1$ .

We call the system  $\mathcal{S}$  a *weighted root system* and  $l$  the *rank* of the system. Denote by  $\text{rank } \mathcal{S}$  the rank of  $\mathcal{S}$ . Let  $W$  be the group generated by the reflections with respect to  $\alpha^{-1}(0)$ 's ( $\alpha \in \Delta$ ). We call  $W$  the *reflection group associated with  $\mathcal{S}$* . Let  $\mathcal{S}_i = (V_i, (\Delta_i, \{m_\alpha \mid \alpha \in \Delta_i\}, \varepsilon_i))$  ( $i = 1, 2$ ) be weighted root systems. If  $\varepsilon_1 = \varepsilon_2$  and if there exists a linear isometry  $\Phi$  of  $V_1$  onto  $V_2$  satisfying

- (i)  $\{\alpha \circ \Phi \mid \alpha \in \Delta_2\} = \Delta_1$ ,
- (ii)  $m_{\alpha \circ \Phi} = m_\alpha$  ( $\alpha \in \Delta_2$ ),

then we say that  $\mathcal{S}_1$  is *isomorphic* to  $\mathcal{S}_2$  and call  $\Phi$  an *isomorphism of  $\mathcal{S}_1$  onto  $\mathcal{S}_2$* . Let  $\mathcal{S} = (V, (\Delta, \{m_\alpha \mid \alpha \in \Delta\}, \varepsilon))$  be a weighted root systems. Let  $S_V(r)$  is the sphere of radius  $r$  centered at the origin in  $V$ . The reflection group  $W$  preserves  $S_V(r)$  invariantly and hence it acts on  $S_V(r)$ . Let  $\Delta_+(\subset \Delta)$  be the positive root system under a lexicographic ordering of  $V^*$  and  $\delta(\in \Delta_+)$  the highest root. See [He] about the definitions of the positive root system and the highest root. Define a positive number  $r_{\mathcal{S}}$  by

$$r_{\mathcal{S}} := \begin{cases} \frac{\pi}{\|\delta\|} & (\text{in case of } \varepsilon = 1) \\ \infty & (\text{in case of } \varepsilon = -1), \end{cases}$$

where  $\|\delta\|$  is the norm of  $\delta$  with respect to the inner product of  $V^*$  induced from  $\langle \cdot, \cdot \rangle$ .

Let  $B_V(r_S)$  be the open ball of radius  $r_S$  centered at the origin in  $V$  and set  $B_S := B_V(r_S) \setminus \{0\}$ . Fix  $r \in (0, r_S)$ . Denote by  $C_W^\infty(S_V(r), V)$  (resp.  $C_W^\infty(S_V(r), B_S)$ ) the space of all  $W$ -equivariant  $C^\infty$ -maps from  $S_V(r)$  to  $V$  (resp.  $B_S$ ) and by  $\text{Imm}_W^\infty(S_V(r), V)$  (resp.  $\text{Imm}_W^\infty(S_V(r), B_S)$ ) the space of all  $W$ -equivariant  $C^\infty$ -immersions of  $S_V(r)$  into  $V$  (resp.  $B_S$ ). For  $\phi \in \text{Imm}_W^\infty(S_V(r), B_S)$ , define a  $W$ -invariant function  $\rho_{S,\phi}$  over  $S_V(r)$  by

$$(1.1) \quad \rho_{S,\phi}(Z) := \sum_{\alpha \in \Delta_+} \frac{m_\alpha \sqrt{\varepsilon} \alpha(\nu(Z))}{\tan(\sqrt{\varepsilon} \alpha(\phi(Z)))} \quad (Z \in S_V(r)),$$

where  $\nu: S_V(r) \rightarrow S_V(1)$  is the Gauss map of  $\phi$  (defined by assigning the outward unit normal vector of  $\phi$  at  $Z$  to each point  $Z$  of  $S_V(r)$ ). Note that, if  $\alpha(\phi(Z)) = 0$ , then we have  $\nu(Z) = \frac{\phi(Z)}{\|\phi(Z)\|}$  and hence  $\frac{\sqrt{\varepsilon} \alpha(\nu(Z))}{\tan(\sqrt{\varepsilon} \alpha(\phi(Z)))}$  implies  $\frac{1}{\|\phi(Z)\|}$ . Define a map  $D_S$  from  $\text{Imm}_W^\infty(S_V(r), B_S)$  to  $C_W^\infty(S_V(r), V)$  by

$$(1.2) \quad D_S(\phi) := \left( \frac{\int_{S_V(r)} (\|\Delta_{\widehat{g}} \phi\| + \rho_{S,\phi}) dv_{\widehat{g}}}{\int_{S_V(r)} dv_{\widehat{g}}} - (\|\Delta_{\widehat{g}} \phi\| + \rho_{S,\phi}) \right) \nu$$

for  $\phi \in \text{Imm}_W^\infty(S_V(r), B_S)$ , where  $g$  is the  $W$ -invariant metric on  $S_V(r)$  induced from  $\langle \cdot, \cdot \rangle$  by  $\phi$ ,  $dv_{\widehat{g}}$  is the volume element of  $\widehat{g}$ ,  $\Delta_{\widehat{g}}$  is the Laplace operator with respect to  $\widehat{g}$  and  $\|\cdot\|$  is the norm of  $(\cdot)$  with respect to  $\langle \cdot, \cdot \rangle$ . We consider the following evolution equation

$$(E_S) \quad \frac{\partial \phi_t}{\partial t} = D_S(\phi_t)$$

for a  $C^\infty$ -family  $\phi_t$  in  $\text{Imm}_W^\infty(S_V(r), B_S)$ . Since  $\|\Delta_{\widehat{g}_t} \phi_t\|$  is the mean curvature of  $\phi_t$ , this evolution equation is interpreted as a modified volume-preserving mean curvature flow equation in  $V$ , where  $\widehat{g}_t$  is the  $W$ -invariant metric on  $S_V(r)$  induced from  $\langle \cdot, \cdot \rangle$  by  $\phi_t$ . Denote by  $\iota_r$  the inclusion map of  $S_V(r)$  into  $V$ . It is clear that  $\iota_r$  is  $W$ -equivariant.

First we prove the following result for the evolution equation  $(E_S)$ .

**Theorem A.** *Let  $\mathcal{S} = (V, (\Delta, \{m_\alpha \mid \alpha \in \Delta\}, \varepsilon))$  be a weighted root system and  $\iota_r$  the inclusion map of  $S_V(r)$  into  $V$ . Then there exists a positive constant  $R_0$  smaller than  $\frac{r_S}{8}$  such that, if  $r < R_0$ , then the solution  $\phi_t$  of the evolution equation  $(E_S)$  satisfying the initial condition  $\phi_0 = \iota_r$  uniquely exists in infinite time and  $\phi_{t_i}$  converges to a  $W$ -equivariant  $C^\infty$ -embedding  $\phi_\infty$  of  $S_V(r)$  into  $B_S$  (in the  $C^\infty$ -topology) as  $i \rightarrow \infty$  for some sequence  $\{t_i\}_{i=1}^\infty$  in  $[0, \infty)$  with  $\lim_{i \rightarrow \infty} t_i = \infty$ . Furthermore,  $\phi_t$  ( $0 \leq t < \infty$ ) remain to be strictly convex and hence so is also  $\phi_\infty$ .*

Let  $G/K$  be an irreducible simply connected rank  $l(\geq 2)$  symmetric space of compact type or non-compact type. A weighted root system of rank  $l$  is defined for  $G/K$  in a natural manner (see Section 3). We call this system the *weighted root system associated with  $G/K$*  and denote it by  $\mathcal{S}_{G/K}$ . Let  $\mathcal{S}_{G/K} = (V, (\Delta, \{m_\alpha \mid \alpha \in \Delta\}, \varepsilon))$ , where we note that

$$\varepsilon = \begin{cases} 1 & \text{(when } G/K \text{ is of compact type)} \\ -1 & \text{(when } G/K \text{ is of non-compact type)}. \end{cases}$$

The vector space  $V$  is identified with the tangent space  $T_{eK}\mathcal{T}$  of a maximal flat totally geodesic submanifold  $\mathcal{T}$  in  $G/K$  through  $eK$ , where  $e$  is the identity element of  $G$ . In the case where  $G/K$  is of compact type,  $\mathcal{T}$  is identified with the quotient space  $V/\Pi$  of  $V$  by a lattice  $\Pi$  in  $V$ , and in the case where  $G/K$  is of non-compact type, it is identified with  $V$  oneself. For convenience, let  $\Pi = \{\mathbf{0}\}$  in the case where  $G/K$  is of non-compact type. Let  $W$  be the reflection group associated with  $\mathcal{S}_{G/K}$ . Then it is shown that  $\Pi$  is  $W$ -invariant. Denote by  $\pi$  the quotient map of  $V$  onto  $V/\Pi = \mathcal{T}$  and  $r(G/K)$  the injective radius of  $G/K$ . Note that  $r(G/K) = \infty$  in the case where  $G/K$  is of non-compact type. It is easy to show that  $r(G/K) = r_{\mathcal{S}_{G/K}}$ .

The main theorem in this paper is as follows.

**Theorem B.** *Let  $\mathcal{S} = (V, (\Delta, \{m_\alpha \mid \alpha \in \Delta\}, \varepsilon))$  and  $\iota_r$  as in Theorem A. Assume that  $r < R_0$  and  $\mathcal{S}$  is isomorphic to one associated to an irreducible simply connected rank  $l(\geq 2)$  symmetric space  $G/K$  of compact type or non-compact type, where  $R_0$  is as in Theorem A. Then the following statements (i) – (iii) hold.*

(i) *The hypersurface  $M := K \cdot \pi(\phi_\infty(S_V(r)))$  in  $G/K$  is a  $K$ -invariant strictly convex spherical soap bubble in  $G/K$ , where  $\phi_\infty$  is as in Theorem A.*

(ii) *Let  $C(\subset V)$  be a Weyl domain (i.e., a fundamental domain of the reflection group  $W$ ) and  $\theta_0$  the element of  $(0, \frac{\pi}{2})$  defined by*

$$\theta_0 := \max_P \max_{Z_1 \in P_{\max}} \max_{Z_2 \in P_{\min, Z_1}} \angle Z_1 \mathbf{0} Z_2 \left( = \max_P \max_{Z_1 \in P_{\min}} \max_{Z_2 \in P_{\max, Z_1}} \angle Z_1 \mathbf{0} Z_2 \right),$$

where  $\angle Z_1 \mathbf{0} Z_2$  denotes the angle between  $\overrightarrow{\mathbf{0}Z_1}$  and  $\overrightarrow{\mathbf{0}Z_2}$ ,  $P$  moves over the Grassmannian of all two-planes in  $V$ ,  $P_{\max}$  (resp.  $P_{\min}$ ) denotes the set of all the maximal (resp. minimal) points of the function  $\rho_{\mathcal{S}, \iota_r}$  over  $S_V(r) \cap \overline{C} \cap P$  ( $\overline{C}$  : the closure of  $C$ ) and  $P_{\max, Z_1}(\subset P_{\max})$  (resp.  $P_{\min, Z_1}(\subset P_{\min})$ ) denotes the set (which is at most two-points set) of all the maximal (resp. minimal) points neighboring  $Z_1$  in  $S_V(r) \cap \overline{C} \cap P$  of the function. Then we have

$$M \subset B\left(\frac{r}{\cos \theta_0}\right) \setminus B(r \cos \theta_0),$$

where  $B(r \cos \theta_0)$  (resp.  $B(\frac{r}{\cos \theta_0})$ ) is the closed geodesic ball of radius  $r \cos \theta_0$  (resp.  $\frac{r}{\cos \theta_0}$ ) in  $G/K$  centered at  $eK$ .

(iii) Let  $\eta_{\max}$  and  $\eta_{\min}$  be the functions defined by

$$\eta_{\max}(s) := \max_{Z \in S_V(1)} \left( \rho_{S, \iota_s}(Z) + \frac{l-1}{s} \right)$$

and

$$\eta_{\min}(s) := \min_{Z \in S_V(1)} \left( \rho_{S, \iota_s}(Z) + \frac{l-1}{s} \right).$$

Then the constant mean curvature  $H_M$  of  $M$  satisfies  $\eta_{\min}(r) \leq H_M \leq \eta_{\max}(r)$ .

*Remark 1.1.* For convenience, denote by  $M(r)$  and  $H(r)$  the soap bubble  $M$  and the mean curvature  $H_M$  as in the statement of Theorem A, respectively. Since the volume of the domain surrounded by  $M(r)$  is strictly increasing (and continuous) with respect to  $r$ , it is shown that  $H(r)$  is strictly decreasing (and continuous) with respect to  $r$ . Easily we can show  $\lim_{r \rightarrow 0} \eta_{\max}(r) = \lim_{r \rightarrow 0} \eta_{\min}(r) = \infty$  and hence  $\lim_{r \rightarrow 0} H(r) = \infty$ . On the other hand, in the case where  $G/K$  is of non-compact type, we have

$$\lim_{r \rightarrow \infty} \eta_{\max}(r) = \max_{Z \in S_V(1)} \sum_{\alpha \in \Delta_+} m_\alpha |\alpha(Z)| (= b_{\max}(G/K))$$

and

$$\lim_{r \rightarrow \infty} \eta_{\min}(r) = \min_{Z \in S_V(1)} \sum_{\alpha \in \Delta_+} m_\alpha |\alpha(Z)| (= b_{\min}(G/K)).$$

Hence we obtain

$$b_{\min}(G/K) \leq \lim_{r \rightarrow \infty} H(r) \leq b_{\max}(G/K).$$

By using (ii) of Theorem B, we can derive the following result.

**Corollary C.** *Under the hypothesis of Theorem B, set*

$$\theta_{G/K} := \max_{(Z_1, Z_2) \in S_V(r) \cap \overline{C}} \angle Z_1 \mathbf{0} Z_2.$$

*Then we have*

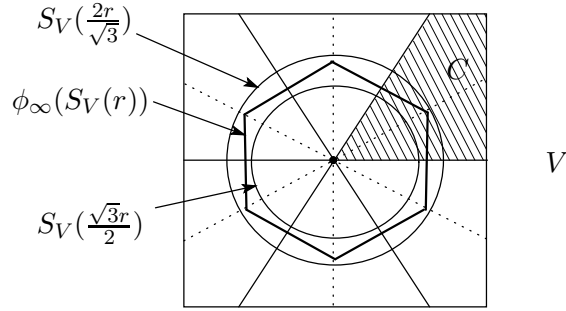
$$M \subset B\left(\frac{r}{\cos \theta_{G/K}}\right) \setminus B(r \cos \theta_{G/K}).$$

In particular, we obtain the following result in the case where  $G/K$  is of rank two.

**Corollary D.** *Under the hypothesis of Theorem B, assume that the rank of  $G/K$  is equal to two. Then we have*

$$M \subset \begin{cases} B\left(\frac{2r}{\sqrt{3}}\right) \setminus B\left(\frac{\sqrt{3}r}{2}\right) & (\triangle : (\mathfrak{a}_2)\text{-type or } (\mathfrak{g}_2)\text{-type}) \\ B(\sqrt{2}r) \setminus B\left(\frac{r}{\sqrt{2}}\right) & (\triangle : (\mathfrak{b}_2)\text{-type}). \end{cases}$$

According to Corollary D, when the root system of  $G/K$  is of type  $\mathfrak{a}_2$ ,  $\phi_\infty(S_V(r))$  is as in Figure 1 for example.



The six corners of  $\phi_\infty(S_V(r))$  are smooth.

The six edges of  $\phi_\infty(S_V(r))$  are curved.

**Figure 1.**

## 2 The volume-preserving mean curvature flow

In this section, we shall recall the definition of the volume-preserving mean curvature flow and the result of N.D. Alikakos and A. Freire ([AF]) for this flow. Let  $M$  be an  $n$ -dimensional compact oriented manifold,  $(\widetilde{M}, \widetilde{g})$  be an  $(n+1)$ -dimensional oriented Riemannian manifold and  $f$  an immersion of  $M$  into  $\widetilde{M}$ . Also, let  $\{f_t\}_{t \in [0, T]}$  be a  $C^\infty$ -family of immersions of  $M$  into  $\widetilde{M}$ , where  $T$  is a positive constant or  $T = \infty$ . Define a map  $F : M \times [0, T) \rightarrow \widetilde{M}$  by  $F(x, t) = f_t(x)$   $((x, t) \in M \times [0, T))$ . Denote by  $\pi_M$  the natural projection of  $M \times [0, T)$  onto  $M$ . For a vector bundle  $E$  over  $M$ , denote by  $\pi_M^* E$  the induced bundle of  $E$  by  $\pi_M$ . Denote by  $g_t$  and  $N_t$  the induced metric and the outward unit normal vector of  $f_t$ , respectively. Also, denote by  $H_t$  the mean curvature of  $f_t$  for  $-N_t$ . Define the function  $H$  over  $M \times [0, T)$  by  $H_{(x, t)} := (H_t)_x$   $((x, t) \in M \times [0, T))$ , the section  $g$  of  $\pi_M^*(T^{(0, 2)}M)$  by  $g_{(x, t)} := (g_t)_x$   $((x, t) \in M \times [0, T))$  and the section  $N$  of  $\pi_M^*(T\widetilde{M})$  by  $N_{(x, t)} := (N_t)_x$   $((x, t) \in M \times [0, T))$ .

$M \times [0, T)$ ), where  $T^{(0,2)}M$  is the tensor bundle of  $(0, 2)$ -type of  $M$  and  $\widetilde{TM}$  is the tangent bundle of  $\widetilde{M}$ . The average mean curvature  $\overline{H}([0, T) \rightarrow \mathbb{R})$  is defined by

$$\overline{H}_t := \frac{\int_M H_t dv_{g_t}}{\int_M dv_{g_t}},$$

where  $dv_{g_t}$  is the volume element of  $g_t$ . G. Huisken ([Hu3]) called the flow  $\{f_t\}_{t \in [0, T)}$  a *volume-preserving mean curvature flow* if it satisfies

$$F_* \left( \frac{\partial}{\partial t} \right) = (\overline{H} - H)N.$$

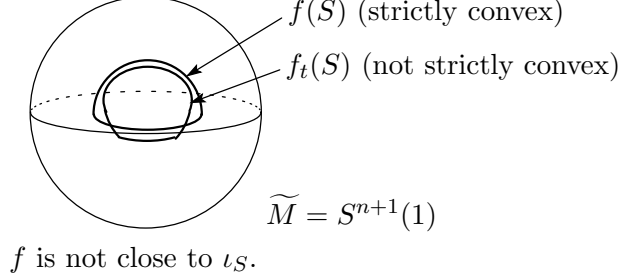
He studied this flow in [Hu3]. Along this flow, the volume of  $(M, g_t)$  decreases but the volume of the domain  $D_t$  surrounded by  $M_t := f_t(M)$  is preserved invariantly. In particular, if  $f_t$ 's are embeddings, then we call  $\{M_t\}_{t \in [0, T)}$  rather than  $\{f_t\}_{t \in [0, T)}$  a volume-preserving mean curvature flow.

Assume that  $\widetilde{M}$  is compact. Let  $S$  be a geodesic sphere in  $\widetilde{M}$  such that, for any point  $p$  of the domain surrounded by  $S$ ,  $S$  lies in a geodesically convex domain of  $p$  (in  $\widetilde{M}$ ), and  $\iota_S$  the inclusion map of  $S$  into  $\widetilde{M}$ . Then, according to Main Theorem of [AF], it is shown that, if  $S$  is a small geodesic sphere of radius smaller than some positive constant among the above geodesic spheres, then for any strictly convex  $C^\infty$ -embedding  $f$  of  $S$  into  $\widetilde{M}$  which is sufficiently close to  $\iota_S$ , there exists the volume-preserving mean curvature flow  $f_t$  starting from  $f$  in infinite time, each  $f_t$  is strictly convex and  $f_{t_i}$  converges to a strictly convex  $C^\infty$ -embedding  $f_\infty$  of constant mean curvature (in  $C^\infty$ -topology) as  $i \rightarrow \infty$  for some sequence  $\{t_i\}_{i=1}^\infty$  with  $\lim_{i \rightarrow \infty} t_i = \infty$ . Furthermore, if all critical points of the scalar curvature functions of  $f_t$ 's are non-degenerate, then  $f_t$  converges to the embedding  $f_\infty$  (in  $C^\infty$ -topology) as  $t \rightarrow \infty$ . Note that the positive constants  $\delta_i$  and  $\varepsilon_i$  ( $i = 0, 1, 2, 3$ ) in the statement of Main Theorem of [AF] are taken as

$$\delta_3 < \delta_2 < \delta_1/2 < \delta_0/4 \quad \text{and} \quad \varepsilon_3 < \varepsilon_2 < \varepsilon_1/2 < \varepsilon_0/4$$

(see P258, 291 and 296 of [AF]) and that  $\delta_0$  and  $\varepsilon_0$  are taken as in P257 of [AF]. By the compactness of  $\widetilde{M}$ , the existenceness of the positive constant  $\delta_0$  in Page 257 of [AF] is assured. Hence the existencenesses of  $\delta_i$  ( $i = 1, 2, 3$ ) also are assured. In the case where  $\widetilde{M}$  is homogeneous, it is clear that their existencenesses are assured even if it is not compact. Hence the statement of Main Theorem of [AF] is valid in the case where  $\widetilde{M}$  is a (not necessarily compact) homogeneous space. Also, we note that the statement of Main Theorem of [AF] does not hold without the assumption that  $f$  is sufficiently close to  $\iota_S$ . For example, in the case where  $f$  is a strictly convex embedding of a small geodesic sphere  $S$  into  $\widetilde{M} = S^{n+1}(1)$  (which is not close to

$\iota_S$ ) as in Figure 2, the volume-preserving mean curvature flow  $f_t$  starting from  $f$  ( $0 \leq t < \infty$ ) does not remain to be strictly convex. This fact is stated in Remarks of Page 38 of [Hu3].



**Figure 2.**

### 3 The mean curvatures of hypersurfaces invariant under the isotropy action

Let  $G/K$  be an irreducible simply connected symmetric space of compact type or non-compact type. In this section, we shall first define the weighted root system associated with  $G/K$ . Set  $n := \dim(G/K) - 1$  and  $l := \text{rank}(G/K)$ . Let  $\mathfrak{g}$  (resp.  $\mathfrak{k}$ ) be the Lie algebra of  $G$  (resp.  $K$ ) and  $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$  the canonical decomposition associated with the symmetric pair  $(G, K)$ . The space  $\mathfrak{p}$  is identified with the tangent space  $T_{eK}(G/K)$  of  $G/K$  at  $eK$ , where  $e$  is the identity element of  $G$ . Take a maximal abelian subspace  $\mathfrak{a}$  of  $\mathfrak{p}$ . Let  $\Delta(\subset \mathfrak{a}^*)$  be the (restricted) root system of the symmetric pair  $(G, K)$  with respect to  $\mathfrak{a}$  and  $\mathfrak{p}_\alpha$  the root space for  $\alpha \in \Delta$ . See [He] about the definitions of these notions. Then we have the following root space decomposition:

$$\mathfrak{p} = \mathfrak{a} \oplus \left( \bigoplus_{\alpha \in \Delta_+} \mathfrak{p}_\alpha \right),$$

where  $\Delta_+$  is the positive root system under some lexicographic ordering of  $\mathfrak{a}^*$ . Set  $m_\alpha := \dim \mathfrak{p}_\alpha$  ( $\alpha \in \Delta$ ). Let  $W$  be the Weyl group of  $G/K$  with respect to  $\mathfrak{a}$  (i.e., the group generated by the reflections with respect to the hyperplanes  $\alpha^{-1}(0)$ 's ( $\alpha \in \Delta$ ) in  $\mathfrak{a}$ ), which acts on  $\mathfrak{a}$ ,  $C(\subset \mathfrak{a})$  be a Weyl domain (i.e., a fundamental domain of the action  $W \curvearrowright \mathfrak{a}$ ) and  $S_{\mathfrak{a}}(r)$  be the sphere of radius  $r$  centered at the origin  $\mathbf{0}$  in  $\mathfrak{a}$ . The isotropy group  $K$  acts on  $G/K$  naturally. This action is called the isotropy action of  $G/K$ . Denote by  $\exp$  the exponential map of  $G$  and  $\text{Exp}$  the exponential map of  $G/K$  at  $eK$ . Set  $\mathcal{T} := \text{Exp}(\mathfrak{a})$ , which is a maximal flat totally geodesic submanifold of  $G/K$ . Note that  $\mathcal{T}$  is identified with the quotient  $\mathfrak{a}/\Pi$  by a  $W$ -invariant lattice  $\Pi$  in  $\mathfrak{a}$  in the case where  $G/K$  is of compact type, and it is identified with  $\mathfrak{a}$  oneself



in the case where  $G/K$  is of non-compact type. Set

$$(3.1) \quad \varepsilon := \begin{cases} 1 & (\text{when } G/K \text{ is of compact type}) \\ -1 & (\text{when } G/K \text{ is of non-compact type}). \end{cases}$$

Then the system  $\mathcal{S} := (\mathfrak{a}, (\Delta, \{m_\alpha \mid \alpha \in \Delta\}, \varepsilon))$  is a weighted root system. We call this system the *weighted root system associated with  $G/K$*  and denote it by  $\mathcal{S}_{G/K}$ .

Next we shall describe explicitly the mean curvatures of  $K$ -invariant hypersurfaces in  $G/K$  in terms of the roots. As a special case, those of geodesic spheres in  $G/K$  are described explicitly. Let  $M^\mathfrak{a}$  be a  $W$ -invariant star-shaped hypersurface (at the origin  $\mathbf{0}$ ) in  $\mathfrak{a}$ . Assume that, in the case where  $G/K$  is of compact type,  $\max_{Z \in M^\mathfrak{a}} d(\mathbf{0}, Z)$  is smaller than the injective radius  $r(G/K)$  of  $G/K$ , where  $d$  is the Euclidean distance of  $\mathfrak{a}$ . Set  $M_\mathcal{T} := \text{Exp}(M^\mathfrak{a})$  and  $M := K \cdot M_\mathcal{T} (= K \cdot (\text{Exp}(M^\mathfrak{a} \cap \overline{C}))$ . Note that  $M_\mathcal{T}$  and  $M$  are hypersurfaces in  $\mathcal{T}$  and  $G/K$ , respectively. Denote by  $N$  the outward unit normal vector field of  $M(\subset G/K)$ , and  $A$  and  $H$  the shape operator and the mean curvature of  $M(\subset G/K)$  for the inward unit normal vector field  $-N$ , respectively. Also, denote by  $A^\mathcal{T}$  and  $H_\mathcal{T}$  those of  $M_\mathcal{T}(\subset \mathcal{T})$  for the inward unit normal vector  $-N|_{M_\mathcal{T}}$ , respectively, and  $\hat{A}$  and  $\hat{H}$  those of  $M^\mathfrak{a}(\subset \mathfrak{a})$  for the inward unit normal vector, respectively. Take any  $Z \in M^\mathfrak{a} \cap C$ . Denote by  $L_Z$  the  $K$ -orbit  $K \cdot \text{Exp } Z$ , which is a principal orbit of the  $K$ -action because  $Z \in C$ . We have

$$T_{\text{Exp } Z} L_Z = (\text{exp } Z)_* \left( \bigoplus_{\alpha \in \Delta_+} \mathfrak{p}_\alpha \right).$$

Denote by  $A^Z$  the shape tensor of  $L_Z(\subset G/K)$ . Take  $X_\alpha \in \mathfrak{p}_\alpha$  ( $\alpha \in \Delta_+$ ). Then we have

$$(3.2) \quad A_{N_{\text{Exp } Z}}^Z((\text{exp } Z)_* X_\alpha) = -\frac{\sqrt{\varepsilon} \alpha((\text{exp } Z)_*^{-1}(N_{\text{Exp } Z}))}{\tan(\sqrt{\varepsilon} \alpha(Z))} (\text{exp } Z)_* X_\alpha$$

by suitably rescaling of the metric of  $G/K$  (see [GT],[K1],[K2]), where  $\frac{1}{\tan(\sqrt{\varepsilon} \alpha(Z))} = 0$  in case of  $\sqrt{\varepsilon} \alpha(Z) = \frac{\pi}{2}$ . In the sequel, we give  $G/K$  this rescaled metric. The vector field  $N|_{L_Z}$  is a  $K$ -equivariant normal vector field along  $L_Z$ . Hence, since the  $K$ -action is a hyperpolar action, it is a parallel normal vector field of  $L_Z$  (see Theorem 5.5.12 of [PT]). Hence, we have

$$(3.3) \quad A_{N_{\text{Exp } Z}}^Z = -A_{\text{Exp } Z}|_{T_{\text{Exp } Z} L_Z},$$

where we note that  $A_{\text{Exp } Z}$  denotes the value of  $A$  at  $\text{Exp } Z$ . In the sequel, we express  $A_{\text{Exp } Z}$  as  $A$  for simplicity. From (3.2) and (3.3), we obtain

$$(3.4) \quad A((\text{exp } Z)_* X_\alpha) = \frac{\sqrt{\varepsilon} \alpha((\text{exp } Z)_*^{-1}(N_{\text{Exp } Z}))}{\tan(\sqrt{\varepsilon} \alpha(Z))} (\text{exp } Z)_* X_\alpha.$$

Take  $X_0 \in \mathfrak{a} \ominus \text{Span}\{N_{\text{Exp } Z}\}$ . Since  $\mathcal{T}$  is totally geodesic in  $G/K$ , we have

$$(3.5) \quad A((\exp Z)_* X_0) = A^T((\exp Z)_* X_0) = (\exp Z)_*(\hat{A}X_0).$$

From (3.4) and (3.5), we obtain

$$(3.6) \quad H_{\text{Exp } Z} = \sum_{\alpha \in \Delta_+} \frac{m_\alpha \sqrt{\varepsilon} \alpha((\exp Z)_*^{-1}(N_{\text{Exp } Z}))}{\tan(\sqrt{\varepsilon} \alpha(Z))} + \hat{H}_Z \quad (Z \in M^{\mathfrak{a}} \cap C).$$

Take any  $Z' \in M^{\mathfrak{a}} \cap \partial C$ , where  $\partial C$  is the boundary of  $C$ . Set  $\Delta_+^{Z'} := \{\alpha \in \Delta_+ \mid \alpha(Z') = 0\}$ . Then it follows from (3.6) and the continuity of  $H$  that

$$(3.7) \quad H_{\text{Exp } Z'} = \sum_{\alpha \in \Delta_+ \setminus \Delta_+^{Z'}} \frac{m_\alpha \sqrt{\varepsilon} \alpha((\exp Z')_*^{-1}(N_{\text{Exp } Z'}))}{\tan(\sqrt{\varepsilon} \alpha(Z'))} + \frac{1}{\|Z'\|} \sum_{\alpha \in \Delta_+^{Z'}} m_\alpha + \hat{H}_{Z'},$$

where we note that  $(\exp Z')_*^{-1}(N_{\text{Exp } Z'}) = \frac{1}{\|Z'\|} Z'$ . The hypersurface  $M$  is of constant mean curvature  $\kappa$  if and only if  $H_{\text{Exp } Z} = \kappa$  ( $Z \in M^{\mathfrak{a}}$ ) because  $M$  is  $K$ -invariant. We consider the case where  $G/K$  is of rank two. Let  $c : [0, b) \rightarrow \mathfrak{a}$  be the curve parametrized by the arclength whose image is equal to  $M^{\mathfrak{a}}$ , where  $b$  is the length of  $M^{\mathfrak{a}}$ . Then it follows from (3.6) that  $M$  is of constant mean curvature  $\kappa$  if and only if the following relation holds:

$$- \sum_{\alpha \in \Delta_+} \frac{m_\alpha \sqrt{\varepsilon} \alpha(c''(s)/\|c''(s)\|)}{\tan(\sqrt{\varepsilon} \alpha(c(s)))} + c''(s) = \kappa \quad (s \in [0, b)).$$

This relation is equivalent to the relation (26) of [Hs, Page 164]. W.Y. Hsiang ([Hs]) derived some facts by using the relation (26).

In particular, we consider the case where  $M$  is a geodesic sphere. Let  $S(r)$  be the geodesic sphere of radius  $r(> 0)$  centered at  $eK$  and  $S_{\mathfrak{a}}(r)$  the sphere of radius  $r$  centered at the origin  $\mathbf{0}$  in  $\mathfrak{a}$ . Assume that  $r$  is smaller than the first conjugate radius of  $G/K$  in the case where  $G/K$  is of compact type. Then we have  $S(r) = K \cdot (\text{Exp}(S_{\mathfrak{a}}(r) \cap \overline{C}))$ . Denote by  $N^r$  the outward unit normal vector of  $S(r)$  and  $H^r$  the mean curvature of  $S(r)$  for  $-N^r$ . Take any  $Z \in S_{\mathfrak{a}}(r) \cap \overline{C}$ . Since  $(N^r)_{\text{Exp } Z} = \frac{1}{r}(\exp Z)_*(Z)$ , it follows from (3.6) and (3.7) that

$$(3.8) \quad (H^r)_{\text{Exp } Z} = \begin{cases} \sum_{\alpha \in \Delta_+} \frac{m_\alpha \sqrt{\varepsilon} \alpha(Z)}{r \tan(\sqrt{\varepsilon} \alpha(Z))} + \frac{l-1}{r} & (Z \in S_{\mathfrak{a}}(r) \cap C) \\ \sum_{\alpha \in \Delta_+ \setminus \Delta_+^{Z'}} \frac{m_\alpha \sqrt{\varepsilon} \alpha(Z)}{r \tan(\sqrt{\varepsilon} \alpha(Z))} \\ + \frac{1}{r} \left( \sum_{\alpha \in \Delta_+^{Z'}} m_\alpha + l - 1 \right) & (Z \in S_{\mathfrak{a}}(r) \cap \partial C). \end{cases}$$

## 4 Proof of Theorem A

In this section, we shall prove Theorem A stated in Introduction. We use the notations in Sections 1-3. Let  $\mathcal{S} = (V, (\Delta, \{m_\alpha \mid \alpha \in \Delta\}, \varepsilon))$ ,  $\phi_t$  and  $\iota_r$  be as in the statement of Theorem A. Assume that there exists a solution  $\phi_t$  ( $t \in [0, T)$ ) of the  $(E_{\mathcal{S}})$  satisfying  $\phi_0 = \iota_r$ . Denote by  $\hat{g}_t$  and  $\nu_t$  the induced metric and the outward unit normal vector of  $\phi_t$ , respectively. Also, denote by  $\hat{h}_t$  and  $\hat{A}_t$  the second fundamental form and the shape operator of  $\phi_t$  for  $-\nu_t$ , respectively. Define the sections  $\hat{g}$  (resp.  $\hat{h}$ ) of  $\pi_{S_V(r)}^*(T^{(0,2)}S_V(r))$  by  $\hat{g}_{(x,t)} := (\hat{g}_t)_x$  (resp.  $\hat{h}_{(x,t)} := (\hat{h}_t)_x$ ) ( $(x, t) \in S_V(r) \times [0, T)$ ), where  $\pi_{S_V(r)}$  denotes the natural projection of  $S_V(r) \times [0, T)$  onto  $S_V(r)$ . Also, denote by  $\hat{\nabla}^t$  the Riemannian connection of  $\hat{g}_t$  and  $\Delta_{\hat{g}_t}$  the Laplace operator with respect to  $\hat{g}_t$ . Define a function  $\hat{r}_t$  ( $t \in [0, T)$ ) over  $S_V(r)$  by

$$\hat{r}_t(Z) := \|\phi_t(Z)\| \quad (Z \in S_V(r))$$

and a diffeomorphism  $\hat{c}_t$  of  $S_V(r)$  by

$$\hat{c}_t(Z) := \frac{r\phi_t(Z)}{\|\phi_t(Z)\|} \quad (Z \in S_V(r)).$$

Then we can derive the following fact for the evolution of  $\hat{r}_t$ .

**Lemma 4.1.** *The functions  $\{\hat{r}_t\}_{t \in [0, T)}$  satisfies the following evolution equation:*

$$(4.1) \quad \frac{\partial \hat{r}}{\partial t} = \left( \frac{\int_{S_V(r)} (\|\Delta_{\hat{g}_t} \phi_t\| + \rho_{\mathcal{S}, \phi_t}) dv_{\hat{g}_t}}{\int_{S_V(r)} dv_{\hat{g}_t}} - (\|\Delta_{\hat{g}_t} \phi_t\| + \rho_{\mathcal{S}, \phi_t}) \right) \\ \times \frac{\hat{r}_t \cdot \|c_{t*}(\text{grad}_t \hat{r}_t)\|}{\sqrt{\|\text{grad}_t \hat{r}_t\|^4 + \hat{r}_t^2 \|c_{t*}(\text{grad}_t \hat{r}_t)\|^2}}.$$

*Proof.* By a simple calculation, we have

$$\begin{aligned} \frac{\partial \hat{r}}{\partial t} &= \langle D_{\mathcal{S}}(\phi_t), \frac{1}{\hat{r}_t} \phi_t \rangle \\ &= \left( \frac{\int_{S_V(r)} (\|\Delta_{\hat{g}_t} \phi_t\| + \rho_{\mathcal{S}, \phi_t}) dv_{\hat{g}_t}}{\int_{S_V(r)} dv_{\hat{g}_t}} - (\|\Delta_{\hat{g}_t} \phi_t\| + \rho_{\mathcal{S}, \phi_t}) \right) \\ &\quad \times \frac{\langle \nu_t, \phi_t \rangle}{\hat{r}_t} \end{aligned}$$

Also, by a simple calculation, we have

$$\begin{aligned} \nu_t = & - \frac{\|\text{grad}_t \hat{r}_t\|^2}{\|c_{t*}(\text{grad}_t \hat{r}_t)\| \sqrt{\|\text{grad}_t \hat{r}_t\|^4 + \hat{r}_t^2 \|c_{t*}(\text{grad}_t \hat{r}_t)\|^2}} \cdot c_{t*}(\text{grad}_t \hat{r}_t) \\ & + \frac{\|c_{t*}(\text{grad}_t \hat{r}_t)\|}{\sqrt{\|\text{grad}_t \hat{r}_t\|^4 + \hat{r}_t^2 \|c_{t*}(\text{grad}_t \hat{r}_t)\|^2}} \cdot \phi_t. \end{aligned}$$

From these relations, we obtain the desired evolution equation.

q.e.d.

The following evolution equation holds for  $\hat{h}_t$ .

**Lemma 4.2.** *The families  $\{\hat{h}_t\}_{t \in [0, \infty)}$  satisfies*

$$\begin{aligned} \frac{\partial \hat{h}}{\partial t} - \Delta_{\hat{g}_t} \hat{h}_t = & \hat{\nabla}^t d(\rho_{\mathcal{S}, \phi_t}) + \text{Tr}(\hat{A}_t^2) \hat{h}_t \\ & + \left( \frac{\int_{S_V(r)} (\|\Delta_{\hat{g}_t} \phi_t\| + \rho_{\mathcal{S}, \phi_t}) dv_{\hat{g}_t}}{\int_{S_V(r)} dv_{\hat{g}_t}} - 2\|\Delta_{\hat{g}_t} \phi_t\| - \rho_{\mathcal{S}, \phi_t} \right) \hat{h}_t(\hat{A}_t \bullet, \bullet). \end{aligned}$$

*Proof.* For simplicity, set  $\hat{H}_t^{\mathcal{S}} := \|\Delta_{\hat{g}_t} \phi_t\| + \rho_{\mathcal{S}, \phi_t}$  and

$$\overline{H}_t^{\mathcal{S}} := \frac{\int_{S_V(r)} (\|\Delta_{\hat{g}_t} \phi_t\| + \rho_{\mathcal{S}, \phi_t}) dv_{\hat{g}_t}}{\int_{S_V(r)} dv_{\hat{g}_t}}.$$

By a simple calculation, we have

$$\frac{\partial \hat{h}}{\partial t} = \hat{\nabla}^t d\hat{H}_t^{\mathcal{S}} + (\overline{H}_t^{\mathcal{S}} - \hat{H}_t^{\mathcal{S}}) \hat{h}_t(\hat{A}_t \bullet, \bullet).$$

Also, by using the Simon's identity, we have

$$\Delta_{\hat{g}_t} \hat{h}_t = \hat{\nabla}^t d\hat{H}_t + \hat{H}_t \hat{h}_t(\hat{A}_t \bullet, \bullet) - \text{Tr}(\hat{A}_t^2) \hat{h}_t.$$

From these relations, we obtain the desired evolution equation.

q.e.d.

Also, we prepare the following lemma, which will be used in the proof of the statement (iii) of Theorem B.

**Lemma 4.3.** *Let  $\overline{H}_t^{\mathcal{S}}$  and  $\hat{H}_t^{\mathcal{S}}$  be as in the proof of Lemma 4.2. The family*

$\{\widehat{H}_t^S\}_{t \in [0, \infty)}$  satisfies the following evolution equation:

$$(4.2) \quad \begin{aligned} \frac{\partial \widehat{H}_t^S}{\partial t} - \Delta_{\widehat{g}_t} \widehat{H}_t^S &= \sum_{\alpha \in \Delta_+} \frac{m_\alpha \sqrt{\varepsilon} \alpha(\phi_{t*}(\text{grad}_{\widehat{g}_t} \widehat{H}_t^S))}{\tan(\sqrt{\varepsilon}(\alpha \circ \phi_t))} \\ &+ (\overline{H}_t^S - \widehat{H}_t^S) \sum_{\alpha \in \Delta_+} \frac{m_\alpha \sqrt{\varepsilon}^2 (\alpha \circ \nu_t)^2 (3 \cos^2(\sqrt{\varepsilon}(\alpha \circ \phi_t)) - 1)}{\sin^2(\sqrt{\varepsilon}(\alpha \circ \phi_t))}. \end{aligned}$$

*Proof.* The family  $\{\widehat{g}_t\}_{t \in [0, \infty)}$  satisfies

$$\frac{\partial \widehat{g}}{\partial t} = 2(\overline{H}_t^S - \widehat{H}_t^S) \widehat{h}_t.$$

From the evolution equation in Lemma 4.2 and this evolution equation, we have

$$\frac{\partial \widehat{H}_t^S}{\partial t} - \Delta_{\widehat{g}_t} \widehat{H}_t^S = \frac{\partial \rho_{S, \phi_t}}{\partial t} + 3(\overline{H}_t^S - \widehat{H}_t^S) \text{Tr}(\widehat{A}_t^2).$$

On the other hand, we have

$$\frac{\partial \rho_{S, \phi_t}}{\partial t} = \sum_{\alpha \in \Delta_+} m_\alpha \left( -\frac{(\overline{H}_t^S - \widehat{H}_t^S) \sqrt{\varepsilon}^2 (\alpha \circ \nu_t)^2}{\sin^2(\sqrt{\varepsilon}(\alpha \circ \phi_t))} + \frac{\sqrt{\varepsilon} \alpha(\phi_{t*}(\text{grad}_{\widehat{g}_t} \widehat{H}_t^S))}{\tan(\sqrt{\varepsilon}(\alpha \circ \phi_t))} \right).$$

From these relations, we can derive the desired evolution equation. q.e.d.

By using Lemmas 4.1 and 4.2, we shall prove Theorem A.

*Proof of Theorem A.* Since  $L_t := \phi_t(S_V(r))$  ( $t \in [0, T)$ ) are  $W$ -invariant, their barycenter are equal to the origin  $\mathbf{0}$  of  $V$ . Hence the barycenter  $\xi(t)$  of  $L_t$  is equal to the origin  $\mathbf{0}$  of  $V$  and the diffeomorphisms  $e(t)$  in the barycentric system (3.1) in Page 288 of [AF] are regarded as the identity transformation of  $S_V(1)$  under the identification of  $T_{\mathbf{0}}V$  and  $V$ . Hence the left-hand sides of the first and the second relations in (3.1) are equal to zero. On the other hand, it is clear that the right-hand sides in the first and the second relations are equal to zero in our setting. Thus the first and the second relations in (3.1) (of [AF]) are trivial. Also, it is easy to show that (4.1) corresponds to the third relation in (3.1) (of [AF]), where we regard  $\widehat{r}_t$  as a function over  $S_V(1)$  under the natural identification of  $S_V(r)$  and  $S_V(1)$ . Here we note that the term  $E$  in the right-hand side of (3.1) (of [AF]) vanishes in our setting because  $E$  is defined by  $E = \langle w, \nu - e \rangle$  in Page 283 of [AF] and, in our setting,  $w$  is equal to  $\mathbf{0}$  by the  $W$ -invariantness of  $\phi_t$ . According to Lemma 3.6 of

[AF], there exists a positive constant  $R_0$  such that, if  $r < R_0$ , then the solution  $\hat{r}_t$  of the evolution equation (4.1) satisfying the initial condition  $\hat{r}_0 = r$  uniquely exists in infinite time. According to the discussion in Page 299 (Line 3 from the bottom)-300 (Line 8) of [AF],  $\hat{r}_{t_i}$  converges to a  $W$ -equivariant  $C^\infty$ -function  $\hat{r}_\infty$  over  $S_V(1)$  (in the  $C^\infty$ -topology) as  $t \rightarrow \infty$  for some sequence  $\{t_i\}_{i=1}^\infty$  in  $[0, \infty)$  with  $\lim_{i \rightarrow \infty} t_i = \infty$ . This fact implies that the solution  $\phi_t$  of  $(E_S)$  satisfying  $\phi_0 = \iota_r$  exists uniquely in infinite time and that  $\phi_{t_i}$  converges to a  $W$ -equivariant  $C^\infty$ -embedding  $\phi_\infty$  of  $S_V(r)$  into  $V$  (in the  $C^\infty$ -topology) as  $i \rightarrow \infty$ . The positive constant  $\delta_3$  (which corresponds to the above  $R_0$ ) in Lemma 3.6 of [AF] is smaller than the positive constant  $\delta_2$  in Lemma 3.2 of [AF],  $\delta_2$  is smaller than the half of the positive constant  $\delta_1 (= \delta_M)$  in Lemma 1.1 of [AF] and  $\delta_1$  is smaller than the half of the positive constant  $\delta_0$  defined in Page 257 of [AF]. Also, according to the definition of  $\delta_0$  (see P257 of [AF]), we see that the positive constant corresponding to  $\delta_0$  is equal to  $\frac{r_S}{2}$  in our setting. Thus  $R_0$  is smaller than  $\frac{r_S}{8}$ . Denote by  $P(\hat{h}_t)$  the right-hand side of the evolution equation in Lemma 4.2. Assume that  $v \in \text{Ker}(\hat{h}_t)_Z$ . We may assume that  $Z \in S_V(r) \cap \overline{C}$  without loss of generality. Then we have

$$(4.3) \quad P(\hat{h}_t)_Z(v, v) = (\hat{\nabla}^t d(\rho_{S, \phi_t}))_Z(v, v).$$

Let  $\gamma$  be the  $\nabla^t$ -geodesic in  $S_V(r)$  with  $\gamma'(0) = v$ . Then we have

$$\begin{aligned} (\hat{\nabla}^t d(\rho_{S, \phi_t}))_Z(v, v) &= \left. \frac{d^2}{ds^2} \right|_{s=0} \rho_{S, \phi_t}(\gamma(s)) \\ &= \sum_{\alpha \in \Delta_+} m_\alpha \left( \frac{\sqrt{\varepsilon} \alpha(\tilde{\nabla}^{\nu_t \circ \gamma} \nu_{t*}(\gamma'(s)))}{\tan(\sqrt{\varepsilon} \alpha(\phi_t(Z)))} - \frac{2\varepsilon \alpha(\phi_t(Z)) \alpha(\nu_{t*}(v))}{\sin^2(\sqrt{\varepsilon} \alpha(\phi_t(Z)))} \right. \\ &\quad - \frac{\sqrt{\varepsilon}^2 \alpha(\nu_t(Z)) \alpha(\tilde{\nabla}^{\phi_t \circ \gamma} \phi_{t*}(\gamma'(s)))}{\sin^2(\sqrt{\varepsilon} \alpha(\phi_t(Z)))} \\ &\quad \left. + \frac{2\sqrt{\varepsilon}^3 \alpha(\phi_{t*}(v))^2 \alpha(\nu_t(Z))}{\sin^2(\sqrt{\varepsilon} \alpha(\phi_t(Z))) \tan(\sqrt{\varepsilon} \alpha(\phi_t(Z)))} \right) \\ &= \sum_{\alpha \in \Delta_+} m_\alpha \left( \frac{\sqrt{\varepsilon} \alpha(\phi_{t*}((\hat{\nabla}_v^t \hat{A}_t)(v))) - \hat{h}_t(\hat{A}_t(v), v) \alpha(\nu_t(Z))}{\tan(\sqrt{\varepsilon} \alpha(\phi_t(Z)))} \right. \\ &\quad - \frac{2\varepsilon \alpha(\phi_t(Z)) \alpha(\phi_{t*}(\hat{A}_t(v)))}{\sin^2(\sqrt{\varepsilon} \alpha(\phi_t(Z)))} - \frac{\varepsilon \alpha(\nu_t(Z))^2 \hat{h}_t(v, v)}{\sin^2(\sqrt{\varepsilon} \alpha(\phi_t(Z)))} \\ &\quad \left. + \frac{2\sqrt{\varepsilon}^3 \alpha(\phi_{t*}(v))^2 \alpha(\nu_t(Z))}{\sin^2(\sqrt{\varepsilon} \alpha(\phi_t(Z))) \tan(\sqrt{\varepsilon} \alpha(\phi_t(Z)))} \right), \end{aligned}$$

where  $\tilde{\nabla}^{\phi_t \circ \gamma}$  (resp.  $\tilde{\nabla}^{\nu_t \circ \gamma}$ ) is the pullback connection of the connection  $\tilde{\nabla}$  of  $V$  by

$\phi_t \circ \gamma$  (resp.  $\nu_t \circ \gamma$ ). Hence, since  $v \in \text{Ker } \widehat{h}_t$ , we obtain

$$(4.4) \quad (\nabla^t d(\rho_{\mathcal{S}, \phi_t}))_Z(v, v) = \sum_{\alpha \in \Delta_+} \frac{m_\alpha \sqrt{\varepsilon}}{\tan(\sqrt{\varepsilon} \alpha(\phi_t(Z)))} \\ \times \left( \alpha(\phi_{t*}((\widehat{\nabla}_v^t \widehat{A}_t)(v))) + \frac{2\varepsilon \alpha(\phi_{t*}(v))^2 \alpha(\nu_t(Z))}{\sin^2(\sqrt{\varepsilon} \alpha(\phi_t(Z)))} \right).$$

According to the proof of Lemma 3.6 of [AF], we may assume that

$$\sup_{t \in [0, \infty)} \|\phi_t(Z)\| < R_0, \quad \sup_{t \in [0, \infty)} \|\phi_{t*}(\widehat{A}_t(v))\| < \varepsilon_1$$

and

$$\sup_{t \in [0, \infty)} \|\phi_{t*}(v)\| < \|v\| + \varepsilon_2$$

for sufficiently small positive constants  $\varepsilon_1$  and  $\varepsilon_2$ , where we note that the statement of Lemma 3.6 is done in the space  $W_{\bullet\bullet}^T$ . See Page 251-252 of [AF] about the definition of  $W_{\bullet\bullet}^T$ , where we note that  $T = \infty$  according to the statement of this lemma. By imitating the discussion in the proof of Lemma 8.3 of [Hu1], it follows from the above second inequality that

$$\sup_{t \in [0, \infty)} \|\phi_{t*}((\widehat{\nabla}_v^t \widehat{A}_t)(v))\| < \varepsilon'_1,$$

where  $\varepsilon'_1$  also is a sufficiently small positive constant because  $\varepsilon_1$  is sufficiently small. Also, we see that  $\alpha(\nu_t(Z)) \geq 0$  ( $t \in [0, \infty)$ ) because the statement of Lemma 3.6 is done in the space  $W_{\bullet\bullet}^\infty$ . Hence, by taking  $R_0$  as a sufficiently small positive constant, we can show that the right-hand side of (4.4) is greater than or equal to zero in all times. Hence we have  $P(\widehat{h}_t)_Z(v, v) \geq 0$  ( $t \in [0, \infty)$ ). Therefore, since  $\phi_0 = \iota_r$  is strictly convex (i.e.,  $\widehat{h}_0 > 0$ ), it follows from the maximum principle that  $\widehat{h}_t > 0$ , that is,  $\phi_t$  is strictly convex for all  $t \in [0, \infty)$  and hence so is also  $\phi_\infty$ . This completes the proof. q.e.d.

## 5 Proof of Theorem B

In this section, we shall prove Theorem B stated in Introduction. We use the notations in Sections 1-4. Let  $\mathcal{S} = (V, (\Delta, \{m_\alpha \mid \alpha \in \Delta\}, \varepsilon))$ ,  $\phi_t$  and  $\iota_r$  be in the statement of Theorem B.

First we shall prove the statement (i) of Theorem B.

*Proof of (i) of Theorem B.* Let  $\phi_t$  ( $t \in [0, \infty)$ ) be the solution of  $(E_{\mathcal{S}})$  satisfying  $\phi_0 = \iota_r$ . The existence of this flow is assured by Theorem A. Define a map  $f_t$  of the

geodesic sphere  $S(r) := K \cdot \pi(S_V(r))$  in  $G/K$  into  $G/K$  by

$$(5.1) \quad f_t(k\pi(Z)) := k\pi(\phi_t(Z)) \quad (k \in K, Z \in S_V(r)).$$

Denote by  $N_t$  the outward unit normal vector field of  $f_t$  and  $A_t$ ,  $H_t$ ,  $\overline{H}_t$  the shape operator, the mean curvature and the average mean curvature of  $f_t$  for the inward unit normal vector field  $-N_t$ , respectively. According to (3.6), we have

$$(5.2) \quad (\overline{H}_t - H_t)N_t \circ \pi|_{S_V(r)} = D_S(\phi_t),$$

where we use the fact that  $\|\Delta_{\widehat{g}_t}\phi_t\|$  is the mean curvature of  $\phi_t$ . Here we note that  $((\overline{H}_t - H_t)N_t)_x \in T_{f_t(x)}\mathcal{T} (= \mathfrak{a} = V)$  ( $x \in \pi(S_V(r))$ ) and hence  $(\overline{H}_t - H_t)N_t \circ \pi|_{S_V(r)}$  is regarded as a map from  $S_V(r)$  to  $V$ . Since  $\phi_t$  ( $t \in [0, \infty)$ ) is the solution of  $(E_S)$  starting from  $\iota_r$ , it follows from (5.2) that  $\{f_t\}_{t \in [0, \infty)}$  is the volume-preserving mean curvature flow starting from the inclusion map  $\iota_{S(r)} : S(r) \hookrightarrow G/K$ . Hence, since  $r < R_0$  by the assumption, it follows from Theorem A that  $\phi_{t_i}$  converges to a strictly convex embedding  $\phi_\infty$  (in  $C^\infty$ -topology) as  $i \rightarrow \infty$  for some sequence  $\{t_i\}_{i=1}^\infty$  in  $[0, \infty)$  with  $\lim_{i \rightarrow \infty} t_i = \infty$  and that  $\phi_t$  ( $0 \leq t < \infty$ ) remain to be strictly convex and hence so is also  $\phi_\infty$ . Let  $f_\infty$  be the map the geodesic sphere  $S(r)$  into  $G/K$  defined as in (5.1) for  $\phi_\infty$  instead of  $\phi_t$ . Since  $\phi_\infty$  is strictly convex, it follows from (3.4) and (3.5) that so is also  $f_\infty$ . Also, since  $\{f_t\}_{t \in [0, \infty)}$  is the volume-preserving mean curvature flow, it follows from Main theorem of [AF] that  $f_\infty$  is of constant mean curvature. This completes the proof. q.e.d.

To prove the statements (ii) and (iii) of Theorem B, we prepare the following lemma.

**Lemma 5.1.** *Let  $Z^{\max}$  (resp.  $Z^{\min}$ ) be one of maximum (resp. minimum) points of  $\rho_{S, \iota_r}$  (hence  $H_0 \circ \pi|_{S_V(r)}$ ). Then the curves  $t \mapsto \phi_t(Z^{\max})$  ( $t \in [0, \infty)$ ) and  $t \mapsto \phi_t(Z^{\min})$  ( $t \in [0, \infty)$ ) are described as*

$$(5.3) \quad \begin{aligned} \phi_t(Z^{\max}) &= \left(1 + \frac{1}{r} \int_0^t (\overline{H}_t - H_t(\pi(Z^{\max}))) dt\right) Z^{\max} & (t \in [0, \infty)), \\ \phi_t(Z^{\min}) &= \left(1 + \frac{1}{r} \int_0^t (\overline{H}_t - H_t(\pi(Z^{\min}))) dt\right) Z^{\min} & (t \in [0, \infty)). \end{aligned}$$

*Proof.* Denote by  $\widehat{g}_t$  and  $\nu_t$  the induced metric and the outward unit normal vector field of  $\phi_t$ , respectively. First we shall calculate  $\frac{\partial \nu}{\partial t}$ . Since  $\langle \nu_t, \nu_t \rangle = 1$ , we have  $\langle \frac{\partial \nu}{\partial t}, \nu_t \rangle = 0$ . Hence  $(\frac{\partial \nu}{\partial t})_Z$  is tangent to  $\phi_t(S_V(r))$  at  $\phi_t(Z)$  for each  $Z \in S_V(r)$ .



Fix  $(Z_0, t_0) \in S_V(r) \times [0, \infty)$ . Let  $\{e_i\}_{i=1}^{l-1}$  be an orthonormal base of  $T_{Z_0}S_V(r)$  with respect to  $(\hat{g}_{t_0})_{Z_0}$  and  $\bar{e}_i$  the tangent vector field of  $S_V(r) \times [0, \infty)$  along  $\{Z_0\} \times [0, \infty)$  defined by  $(\bar{e}_i)_{(Z_0, t)} := (e_i)_{(Z_0, t)}^L$  ( $(Z_0, t) \in \{Z_0\} \times [0, \infty)$ ), where  $(e_i)_{(Z_0, t)}^L$  is the horizontal lift of  $e_i$  to  $(Z_0, t)$  (with respect to the natural projection of  $S_V(r) \times [0, \infty)$  onto  $S_V(r)$ ). Then we have

$$\begin{aligned}
(5.4) \quad & \left( \frac{\partial \nu}{\partial t} \right)_{(Z_0, t_0)} = \sum_{i=1}^{l-1} \left\langle \left( \frac{\partial \nu}{\partial t} \right)_{(Z_0, t_0)}, \phi_{t_0*}(e_i) \right\rangle \phi_{t_0*}(e_i) \\
& = - \sum_{i=1}^{l-1} \left\langle (\nu_{t_0})_{Z_0}, \left( \frac{\partial \phi_{t*}(e_i)}{\partial t} \right)_{(Z_0, t_0)} \right\rangle \phi_{t_0*}(e_i) \\
& = - \sum_{i=1}^{l-1} \left\langle (\nu_{t_0})_{Z_0}, \frac{\partial}{\partial t} (\bar{e}_i \phi) \Big|_{t=t_0} \right\rangle \phi_{t_0*}(e_i) \\
& = - \sum_{i=1}^{l-1} \left\langle (\nu_{t_0})_{Z_0}, e_i \left( \frac{\partial \phi}{\partial t} \Big|_{t=t_0} \right) \right\rangle \phi_{t_0*}(e_i) \\
& = - \sum_{i=1}^{l-1} \langle (\nu_{t_0})_{Z_0}, e_i (\bar{H}_{t_0} - (H_{t_0} \circ \pi|_{S_V(r)})) (\nu_{t_0})_{Z_0} \rangle \phi_{t_0*}(e_i) \\
& = \sum_{i=1}^{l-1} e_i (H_{t_0} \circ \pi|_{S_V(r)}) \phi_{t_0*}(e_i) \\
& = \sum_{i=1}^{l-1} \hat{g}_{t_0}((\text{grad}_{\hat{g}_{t_0}}(H_{t_0} \circ \pi|_{S_V(r)}))_{Z_0}, e_i) \phi_{t_0*}(e_i) \\
& = (\phi_{t_0})_*((\text{grad}_{\hat{g}_{t_0}}(H_{t_0} \circ \pi|_{S_V(r)}))_{Z_0}),
\end{aligned}$$

where we use  $[\frac{\partial}{\partial t}, \bar{e}_i] = 0$ .

Now we shall derive (5.3) in terms of (5.4). It is clear that there exists a  $C^\infty$ -curve  $t \mapsto Z_t^{\max}$  ( $t \in [0, \varepsilon)$ ) such that  $Z_0^{\max} = Z^{\max}$  and that  $Z_t^{\max}$  is a maximum point of  $H_t \circ \pi|_{S_V(r)}$  for each  $t \in [0, \varepsilon)$ , where  $\varepsilon$  is a positive constant. Similarly, there exists a  $C^\infty$ -curve  $t \mapsto Z_t^{\min}$  ( $t \in [0, \hat{\varepsilon})$ ) such that  $Z_0^{\min} = Z^{\min}$  and that  $Z_t^{\min}$  is a minimum point of  $H_t \circ \pi|_{S_V(r)}$  for each  $t \in [0, \hat{\varepsilon})$ , where  $\hat{\varepsilon}$  is a positive constant. According to (5.3), for each  $t_0 \in [0, \varepsilon)$ ,  $\frac{\partial \|\phi_t\|}{\partial t}|_{(Z_{t_0}^{\max}, t_0)} < 0$  and  $Z_{t_0}^{\max}$  is a minimum point of  $Z \mapsto \frac{d\|\phi_t\|}{dt}|_{(Z, t_0)}$ . On the other hand, according to (5.4), we have  $\frac{d(\nu_t)}{dt}|_{Z_t^{\max}} = 0$ , that is,  $(\nu_t)_{Z_t^{\max}} = (\nu_0)_{Z^{\max}}$  ( $t \in [0, \varepsilon)$ ). From these facts, we can derive that  $\phi_t(Z_t^{\max}) = \lambda_1(t)Z^{\max}$  ( $t \in [0, \varepsilon)$ ) for some positive function  $\lambda_1$  over  $[0, \varepsilon)$  and that  $Z_t^{\max} = Z^{\max}$  ( $t \in [0, \varepsilon)$ ) holds. Similarly, we can show that  $\phi_t(Z_t^{\min}) = \lambda_2(t)Z^{\min}$  ( $t \in [0, \hat{\varepsilon})$ ) for some positive function  $\lambda_2$  over  $[0, \hat{\varepsilon})$  and that

$Z_t^{\min} = Z^{\min}$  ( $t \in [0, \widehat{\varepsilon})$ ) holds. Hence it follows from (5.3) that

$$\frac{\partial \lambda_1}{\partial t} = \frac{1}{r} (\overline{H}_t - H_t(\pi(Z^{\max}))) .$$

and

$$\frac{\partial \lambda_2}{\partial t} = \frac{1}{r} (\overline{H}_t - H_t(\pi(Z^{\min}))) .$$

Therefore we can derive

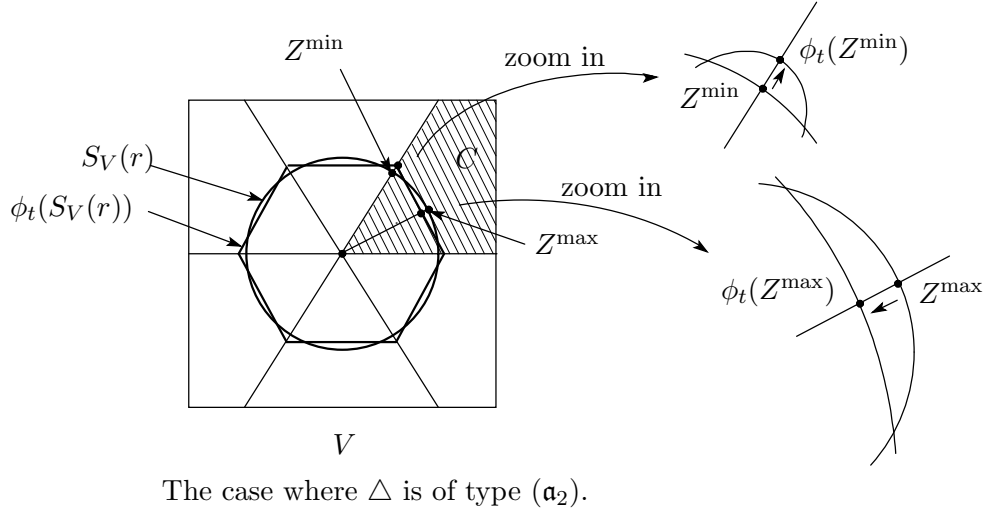
$$\begin{aligned} \phi_t(Z^{\max}) &= \left( 1 + \frac{1}{r} \int_0^t (\overline{H}_t - H_t(\pi(Z^{\max}))) dt \right) Z^{\max} & (t \in [0, \varepsilon)) \\ \phi_t(Z^{\min}) &= \left( 1 + \frac{1}{r} \int_0^t (\overline{H}_t - H_t(\pi(Z^{\min}))) dt \right) Z^{\min} & (t \in [0, \widehat{\varepsilon})). \end{aligned}$$

It is easy to show that these relations hold over  $[0, \infty)$ . This completes the proof.

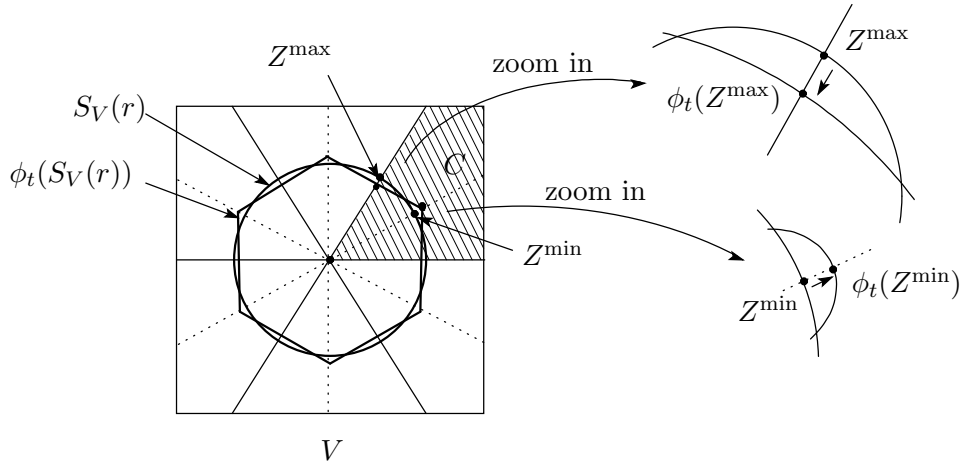
q.e.d.

According to this lemma, in the case where  $\Delta$  is of type  $(\mathfrak{a}_2)$ , the flow  $\phi_t(S_V(r))$  is as in Figure 3 or 4 for example. By using this lemma, we prove the statements (ii) and (iii) of Theorem B.

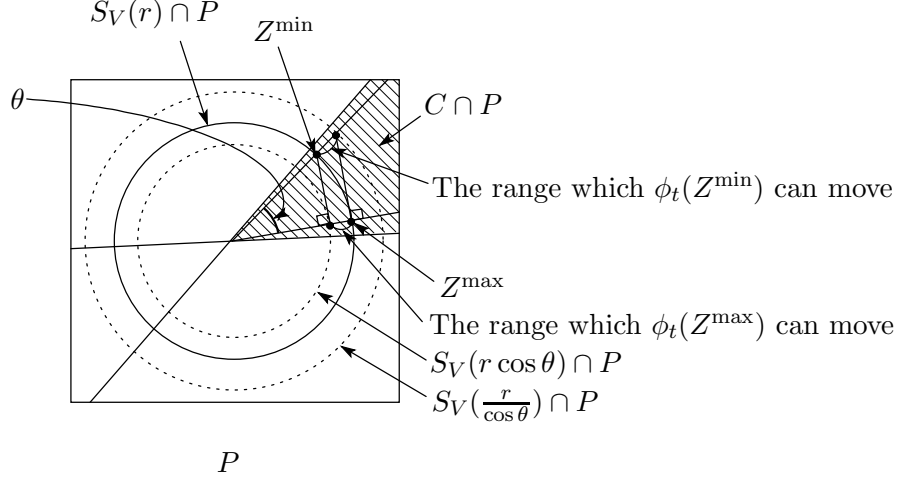
*Proof of (ii) and (iii) Theorem B.* Let  $P$  be a two-plane in  $V$  with  $P_{\max} \neq \emptyset$  and  $P_{\min} \neq \emptyset$ . Take  $Z^{\max} \in P_{\max}$  and  $Z^{\min} \in P_{\min, Z^{\max}}$ . According to Lemma 5.1,  $\phi_t(Z^{\max})$  and  $\phi_t(Z^{\min})$  are as in (5.3). Let  $\theta$  be the angle between  $\overrightarrow{\mathbf{0}Z^{\max}}$  and  $\overrightarrow{\mathbf{0}Z^{\min}}$ . From the convexity of  $\phi_t(S_V(r))$ , we can derive that the part between  $\phi_t(Z^{\max})$  and  $\phi_t(Z^{\min})$  of the curve  $\phi_\infty(S_V(r)) \cap P \cap C$  is included by  $(B_V(\frac{r}{\cos \theta}) \setminus B_V(r \cos \theta)) \cap P \cap C$  (see Figure 5). From this fact and the definition of  $\theta_0$ , we can derive  $M \subset B(\frac{r}{\cos \theta_0}) \setminus B(r \cos \theta_0)$ . Thus the statement (ii) of Theorem B follows. Since  $r$  is a sufficiently small positive constant smaller than  $R_0$ ,  $\max_{S_V(r)} \widehat{r}_t$  is sufficiently small for all  $t \in [0, \infty)$ . Hence we have  $3 \cos^2(\sqrt{\varepsilon}(\alpha \circ \phi_t)) \geq 1$  for all  $t \in [0, \infty)$ . Therefore, according to the maximum principle, it follows from the evolution equation (4.2) for  $\widehat{H}_t^S$  that  $\min_{S_V(r)} \widehat{H}_0^S \leq \widehat{H}_t^S \leq \max_{S_V(r)} \widehat{H}_0^S$ , which implies that  $\min_{S(r)} H^r \leq H_t \leq \max_{S(r)} H^r$  and hence  $\min_{S(r)} H^r \leq H_M \leq \max_{S(r)} H^r$ . On the other hand, according to (3.8), we have  $\eta_{\min}(r) \leq H^r \leq \eta_{\max}(r)$ . Therefore we obtain  $\eta_{\min}(r) \leq H_M \leq \eta_{\max}(r)$ .  
q.e.d.



**Figure 3.**



**Figure 4.**



**Figure 5.**

*Proof of Corollary C.* Let  $\theta_0$  be as in the statement (ii) of Theorem B. Clearly we have  $\theta_0 \leq \theta_{G/K}$ . Hence we obtain the desired inclusion relation from (ii) of Theorem B. q.e.d.

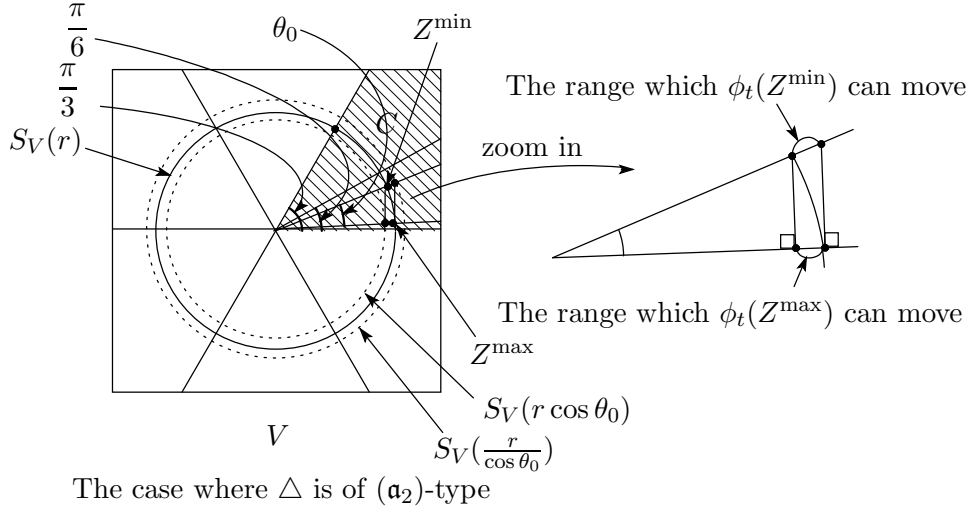
*Proof of Corollary D.* Since  $G/K$  is of rank two, we have

$$\theta_{G/K} = \begin{cases} \frac{\pi}{3} & \text{(when the root system of } G/K \text{ is of type } \mathfrak{a}_2) \\ \frac{\pi}{4} & \text{(when the root system of } G/K \text{ is of type } \mathfrak{b}_2) \\ \frac{\pi}{6} & \text{(when the root system of } G/K \text{ is of type } \mathfrak{g}_2). \end{cases}$$

In the case where  $\Delta$  is of type  $(\mathfrak{a}_2)$ , it is symmetric for two simple roots with considering the multiplicities. Hence, in this case, it follows that  $\theta_0$  in Theorem B are smaller than or equal to the half of  $\theta_{G/K}$  ( $=$  the half of the length of  $S_V(1) \cap \overline{C}$ ), that is, they are smaller than or equal to  $\frac{\pi}{6}$  (see Figure 6). Therefore, from (ii) of Theorem B, we obtain

$$M \subset \begin{cases} B\left(\frac{2r}{\sqrt{3}}\right) \setminus B\left(\frac{\sqrt{3}r}{2}\right) & (\Delta : (\mathfrak{a}_2)\text{-type or } (\mathfrak{g}_2)\text{-type}) \\ B(\sqrt{2}r) \setminus B\left(\frac{r}{\sqrt{2}}\right) & (\Delta : (\mathfrak{b}_2)\text{-type}). \end{cases}$$

q.e.d.



**Figure 6.**

## References

- [AF] N.D. Alikakos and A. Freire, The normalized mean curvature flow for a small bubble in a Riemannian manifold, *J. Differential Geom.* **64** (2003) 247-303.
- [CM] E. Cabezas-Rivas and V. Miquel, Volume preserving mean curvature flow in the hyperbolic space, *Indiana Univ. Math. J.* **56** (2007) 2061-2086.
- [GT] O. Goertsches and G. Thorbergsson, On the Geometry of the orbits of Hermann actions, *Geom. Dedicata* **129** (2007) 101-118.
- [He] S. Helgason, *Differential geometry, Lie groups and symmetric spaces*, Academic Press, New York, 1978.
- [Hs] W.Y. Hsiang, On soap bubbles and isoperimetric regions in non-compact symmetric spaces, I, *Tohoku Math. J.* **44** (1992) 151-175.
- [HH] W.T. Hsiang and W.Y. Hsiang, On the uniqueness of isoperimetric solutions and imbedded soap bubbles in non-compact symmetric spaces, I, *Invent. Math.* **98** (1989) 39-58.
- [Hu1] G. Huisken, Flow by mean curvature of convex surfaces into spheres, *J. Differential Geom.* **20** (1984) 237-266.
- [Hu2] G. Huisken, Contracting convex hypersurfaces in Riemannian manifolds by their mean curvature, *Invent. math.* **84** (1986) 463-480.
- [Hu3] G. Huisken, The volume preserving mean curvature flow, *J. reine angew. Math.* **382** (1987) 35-48.
- [K1] N. Koike, Actions of Hermann type and proper complex equifocal submanifolds, *Osaka J. Math.* **42** (2005) 599-611.
- [K2] N. Koike, Collapse of the mean curvature flow for equifocal submanifolds, *Asian J. Math.* **15** (2011) 101-128.

[PT] R.S. Palais and C.L. Terng, Critical point theory and submanifold geometry, Lecture Notes in Math. **1353**, Springer, Berlin, 1988.

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